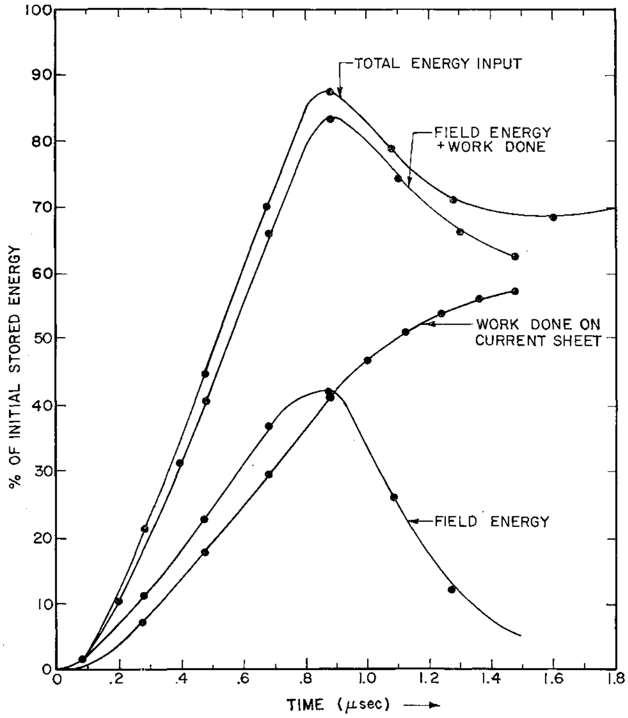

Fig. 5 Energy inventory for the accelerator.

forms of internal energy.[3] Some of the energy in thermal motion of the ions and electrons may subsequently appear in directed energy as a result of expansion near the muzzle of the gun. However, only a small amount of thermal energy can possibly be reclaimed for the following reasons. The plasma density is about 10^{16} cm^{-3} and, in a plasma of high Z material like nitrogen, the electron temperature will be radiation-limited to about 5–10 ev[2,4,5]; consequently the thermal energy stored in the electrons is small compared with the average ion energy in directed motion ($\approx$700 ev). The ions will rapidly lose their thermal energy by coulomb collisions with the cold electrons and will have little left when they reach the muzzle (the e folding time is $\approx$0.25 μsec for 5 ev electrons at a density of 10^{16} cm^{-3}).[6]

From the preceding arguments about 30% of the initial stored energy is expected to appear in the exhaust. (Ionization energy is insignificant compared with the directed energy.) A calorimeter in the exhaust collected only 15% of the initial stored energy; in addition, measurements in the exhaust plasma[7] showed that the average velocity was 6 cm/μsec compared with the sheet speed of 10 cm/μsec. When the barrel length was reduced from 16 to 9 cm, so that the current sheet reached the muzzle at the time of voltage reversal, the calorimetric efficiency increased to 30% and the average velocity of the exhaust plasma equaled the sheet speed. These two results suggest that increased wall losses occur if the current sheet is still inside the barrels when the current decays.

By improving the match between the load and the energy source and using larger diameter barrels, efficiences up to 45% have been obtained, but it is difficult to see how an over-all efficiency greater than 50% can be achieved if most of the mass accelerated is picked up by the current sheet while it is traveling at its terminal velocity. Higher efficiencies are theoretically possible if a plasma slug of constant mass is accelerated.[3,8,9] However experiment has indicated that the current sheet in a coaxial gun tends to become unstable unless it continually sweeps up gas inside the gun.[10] Such an instability may make the simple slug model impossible to achieve. With a gas-filled gun the slug model may be appropriate after the plasma reaches the muzzle if appreciable acceleration takes place outside the gun where the neutral gas density is very low; in this case the efficiency could exceed 50%.

An alternative approach to the problem of improving efficiency in the pulsed coaxial rail accelerator is to continually feed propellant into the gun at the breech. If propellant is fed in at a sufficient rate then the current sheet should remain stationary while plasma accelerates through it. This is a deflagration-type phenomenon and high efficiencies can be expected; this expectation is encouraged by the performance of thermo-ionic arc jets.[1]

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A Solution of the Base Pressure Problem Applicable to Transient External Flows

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Wake flows have been of interest for a considerable time and both steady-flow cases and quasi-steady aspects of transient cases have been investigated.[1–4] All of these investigations related the dynamics of the external isentropic stream and the dissipative mechanism of the jet mixing region.

Inspection of the system of equations describing the mechanism of wake flow under steady condition[2] does not give much encouragement for extending it formally to include all possible additional nonsteady terms. A quasi-steady solution has been made that shows good agreement with limited experimental checks.[4] This quasi-steady theory contains several limitations that restrict its application to wake flows involving time-dependent, external stream properties.

Referring to a simple backstep geometry (see Fig. 1), the quasi-steady theory presented previously[4] assumes that the transient properties of the approaching stream number M_{1a} (or Crocco number C_{1a}), stagnation pressure P_{01a}, and the stagnation temperature T_{01a} are varying slowly enough such that the volume of the wake is not materially effected (θ_{2a} remains substantially constant). For the same reason it is

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assumed that flow is isoenergetic (stagnation temperature constant throughout flow region at any instant) and that the adjustment to pressure waves is essentially instantaneous. The work of Smyrl,[5] Miles,[6] and Battin[7] suggests that adjustment times for short time pressure and stagnation temperature changes are very nearly

$$\Delta t_a = x/c_{2a}$$

where x is the length of the mixing region and c_{2a} is the acoustic velocity of the external stream. Thus the quasi-steady assumptions of pressure and isoenergetic flow should be applicable for highly transient external flows. For cases where the approaching flow properties are highly transient, the assumption is not valid that the wake volume remains substantially constant and, therefore, its variation with M_{1a} must be accounted for in the analysis.

The following quasi-steady analysis is for the simple backstep geometry (Fig. 1) and is restricted to cases where the boundary layer of the approaching stream is negligible or where the mixing region is long enough so that the mixing profile is fully developed and, therefore, the "restricted theory"[8] can be used. The assumptions of isoenergetic flow and essentially instantaneous pressure adjustment, as mentioned previously, are used. For nonsteady flow, the continuity equation is

$$\frac{\partial w}{\partial t} = -\int_A \rho u \cdot dA \tag{1}$$

where u is velocity. Applying Eq. (1) to the wake region where the mass w is

$$w = \rho_b V \text{ lbm/unit width} \tag{2}$$

ρ_b is the bulk density and the wake volume V is

$$V = \tfrac{1}{2}H^2 \cot\theta_{2a} \tag{3}$$

Using the equation of state of a perfect gas, the familiar isentropic relations, and the definition of Crocco number

$$C = u/\left[\frac{2k}{k-1}RT_0\right]^{1/2}$$

Eqs. (2) and (3) can be expressed as

$$w = (P_{01a}/RT_{01a})(1 - C_{2a}{}^2)^{k/k-1}(\tfrac{1}{2}H^2 \cot\theta_{2a}) \tag{4}$$

where the subscripts 1 and 2 refer to the flow before and after the expansion.

The expression for the mass in the wake is differentiated with respect to time treating all the flow parameters as time dependent

$$\frac{dw}{dt} = \frac{1}{2}\frac{H^2}{RT_{01a}}(1 - C_{2a}{}^2)^{k/k-1}\left\{\cot\theta_{2a}\left(\frac{dP_{01a}}{dt} - \frac{P_{01a}}{T_{01a}}\frac{dT_{01a}}{dt}\right) - P_{01a}\left[(1 + C_{2a}{}^2)^{-1}\left(\frac{k}{k-1}\right)\cot\theta_{2a} \times \frac{dC_{2a}{}^2}{dt} - \csc^2\theta_{2a}\frac{d\theta_{2a}}{dt}\right]\right\} \tag{5}$$

From Prandtl-Meyer flow relations

$$d\theta = \frac{(M^2-1)^{1/2}\,dM^2}{2M^2[1 + (k - 1/2)M^2]} = \frac{1}{2C^2}\left[\left(\frac{2}{k-1}\right)\left(\frac{C^2}{1-C^2}\right) - 1\right]^{1/2} dC^2$$

and putting it in terms of the problem

$$\frac{d\theta_{2a}}{dt} = \frac{d\theta_{2a}}{dC_{2a}{}^2} \times \frac{dC_{2a}{}^2}{dt} = \frac{1}{2C_{2a}{}^2} \times \left[\left(\frac{2}{k-1}\right)\left(\frac{C_{2a}{}^2}{1-C_{2a}{}^2}\right) - 1\right]^{1/2}\frac{dC_{2a}{}^2}{dt} \tag{6}$$

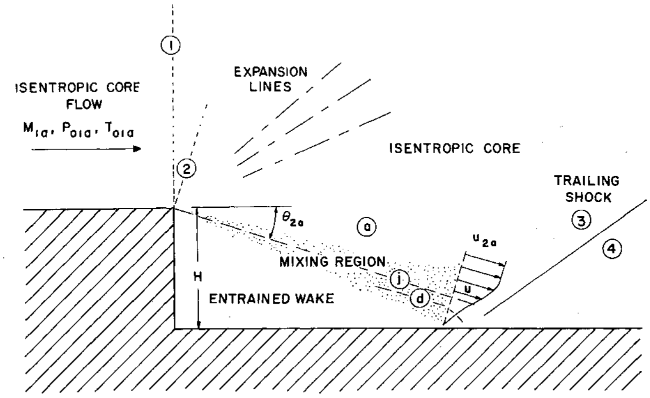

Fig. 1. Backstep flow model.

Substituting (6) into (5) yields

$$\frac{dw}{dt} = \frac{1}{2}\frac{H^2}{RT_{01a}}(1 - C_{2a}{}^2)^{k/k-1}\left(\cot\theta_{2a}\left(\frac{dP_{01a}}{dt} - \frac{P_{01a}}{T_{01a}}\frac{dT_{01a}}{dt}\right) - P_{01a}\left\{(1 - C_{2a}{}^2)^{-1}\left(\frac{k}{k-1}\right)\cot\theta_{2a} + \frac{\csc^2\theta_{2a}}{2C_{2a}{}^2}\left[\left(\frac{2}{k-1}\right)\left(\frac{C_{2a}{}^2}{1-C_{2a}{}^2}\right) - 1\right]^{1/2}\right\}\frac{dC_{2a}{}^2}{dt}\right) \tag{7}$$

which gives an expression for the rate of change of mass in the wake region which in turn must equal the flow out of the wake through the recompression region located between cross sections 3 and 4 of Fig. 1:

$$\frac{dw}{dt} = \int_{yj}^{yd} \rho u\, dy = G_d \tag{8}$$

Equation (8) describes the flow per unit width between the jet boundary streamline, subscript j, originating at the separation point and the discriminating streamline, subscript d, which satisfies the escape criterion (8)

$$P_{03a}/P_3 = P_4/P_3 = (1 - C_{3d}{}^2)^{-k/k-1} \tag{9}$$

where

$$P_4/P_3 = P_4/P_3(M_{2a}, \theta_{4a} - \theta_{2a}, k) \tag{10}$$

is determined by the usual oblique shock relations applied to the adjacent freestream at the end of the wake. For a given expansion angle θ_{2a}, C_{3d} can be calculated.

The "restricted" theory (8) gives the velocity distribution for the flow in the jet mixing region, so that in dimensionless form

$$\psi = u/u_{3a} \qquad \eta = \sigma(y/x) \qquad \psi(\eta) = \tfrac{1}{2}(1 + \operatorname{erf}\eta) \tag{11}$$

where σ is the mixing parameter and, from Ref. 9, $\sigma = 12 + 2.75\, M_a$, and subscript a refers to freestream conditions.

For a constant pressure-mixing process and isoenergetic flow, the mass flow between the j and d streamlines can be expressed by

$$G_d = \rho_{2a}u_{2a}(x/\sigma)(1 - C_{2a}{}^2)(I_{1,d} - I_{1,j}) \tag{12}$$

where

$$I_1 = I_1(C_{2a}{}^2, \eta) = \int_\infty^\eta \frac{\psi}{1 - C_{2a}{}^2\psi^2} \tag{13}$$

is available in tabulated form (10). C_{3d} is determined using Eqs. (9) and (10), $\psi_{3d} = C_{3d}/C_{2a}$, where $C_{2a}, = C_{3a}$ for $P_2 = P_3$ and obtains η_d with Eqs. (11). $I_{1,j}$ is a function only of C_{2a}

and is tabulated in Ref. 10. For the adjacent isentropic free jet

$$\rho_{2a}u_{2a} = \frac{P_{01a}}{(T_{01a})^{1/2}}\left[\frac{R}{kg}\frac{2}{k-1}C_{2a}^2(1 - C_{2a}^2)^{2/k-1}\right]^{1/2} \quad (14)$$

and from the geometry

$$x = H/\sin\theta_{2a} \quad (15)$$

Combining Eqs. (12, 14, and 15) yields

$$G_d = \frac{HC_{2a}P_{01a}}{\sigma\sin\theta_{2a}}\left(\frac{2kg}{(k-1)RT_{01a}}\right)^{1/2}(1 - C_{2a}^2)^{k/k-1}(I_{1,d} - I_{1,j}) \quad (16)$$

Finally, combining Eqs. (5) and (16), one obtains

$$\frac{d}{dt}C_{2a}^2 = \left[\frac{k}{k-1}\frac{\cos\theta_{2a}}{2C_{2a}^2}\left(\frac{2}{k-1}\frac{C_{2a}^2}{1-C_{2a}^2} - 1\right)^{1/2}\right]^{-1} \times \left[\frac{\cos\theta_{2a}}{P_{01a}}\left(\frac{dP_{01a}}{dt} - \frac{P_{01a}}{T_{01a}}\frac{dT_{01a}}{dt}\right) - \frac{2C_{2a}}{H\sigma}\left(\frac{2kg\,RT_{01a}}{k-1}\right)^{1/2}(I_{1,d} - I_{1,j})\right] \quad (17)$$

Integration of Eq. (17) for given forcing functions $P_{01a}(t)$ and $T_{01a}(t)$ with $M_{1a}(t)$ used to get θ_{2a} from the steady-state solution, yields $C_{2a}(t)$. Using the isentropic relation for pressure ratio, in terms of Crocco number, finally the base pressure ratio

$$(P_b/P_{01a})(t) = [1 - C_{2a}^2(t)]^{k/k-1}$$

is determined.

Integration of Eq. (17) lends itself to a simple difference solution and has been found to be relatively insensitive to the size of the steps. The last term of Eq. (17) dominates the solution and confirms the tacit assumption that the pumping action of the mixing region plays the major role in the response of the wake to external disturbances. The "restricted theory" has been used since there is a general insufficiency of information on approaching boundary layers and has given good results in the solution of base pressure problems (8).

A quasi-steady theory has been developed that adequately describes the pressure response of a two-dimensional wake to highly transient external flows and that should give approximate results for step changes in the external flow.

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Radiative Heat-Flux Potential

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THE reduction of the integrodifferential equation governing radiative transfer to a pure differential equation has received considerable attention. This is usually accomplished by taking moments of the radiative transport equation (in terms of the specific intensity) for a gray gas in local thermodynamic equilibrium, expanding the specific intensity in spherical harmonics, and keeping only the first term. This assumes that the radiation obeys the Milne-Eddington directional approximation. The system of moment equations is thus closed, and the resulting equation for the first moment (radiative heat flux) is[1, 2]

$$\nabla(\nabla\cdot\mathbf{q}_{\mathrm{rad}}) - 3\,\mathbf{q}_{\mathrm{rad}} + 4\pi\nabla B = 0 \quad (1)$$

where $\mathbf{q}_{\mathrm{rad}}$ is the radiative heat-flux vector, $B = \sigma T^4/\pi$ (σ is the Stefan-Boltzmann constant, and T the temperature), and all distances are measured in terms of optical paths (the absorption coefficient has been absorbed in the ∇). It seems not to have been noticed that Eq. (1) implies

$$\nabla \times \mathbf{q}_{\mathrm{rad}} = 0 \quad (2)\dagger$$

Hence, $\mathbf{q}_{\mathrm{rad}} = \nabla\psi$, and $\nabla(\nabla^2\psi - 3\psi + 4\pi B) = 0$, or integrating, $\nabla^2\psi - 3\psi + 4\pi B = \mathrm{const.}$ The constant of integration may be absorbed in the heat-flux potential ψ, so that

$$\nabla^2\psi - 3\psi = -4\pi B \quad (3)$$

an inhomogeneous Helmholtz equation. The formal solution to Eq. (3) is given by Bateman,[3] and it shows that the radiative heat-flux potential is equal to the integral over-all space of the local emission (B) exponentially attenuated between the parameter point and the argument point in addition to the contributions from the boundary. The divergence of the heat-flux vector equals $3\psi - 4\pi B$, and this bears a remarkable resemblance to the general form for a gray gas in local thermodynamic equilibrium derived by Goulard.[4] However, the difference in the details of the exponential attenuation suggests the difference between the exponential kernel and the exponential integral kernel in the one-dimensional differential approximation.[5]

In conclusion, we have reduced the solution of a vector differential equation with three components [Eq. (1)] to the solution of a single scalar differential equation of standard form [Eq. (3)] by the introduction of the radiative heat-flux

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